

Linear Quadratic Optimal Control Problems for Mean-Field Backward Stochastic Differential Equations

Xun Li*, Jingrui Sun[†], and Jie Xiong[‡]

October 11, 2016

Abstract: This paper is concerned with linear quadratic optimal control problems for mean-field backward stochastic differential equations (MF-BSDEs, for short) with deterministic coefficients. The optimality system, which is a linear mean-field forward-backward stochastic differential equation with constraint, is obtained by a variational method. By decoupling the optimality system, two coupled Riccati equations and an MF-BSDE are derived. It turns out that the coupled two Riccati equations are uniquely solvable. Then a complete and explicit representation is obtained for the optimal control.

Key words: linear quadratic optimal control, mean-field backward stochastic differential equation, Riccati equation, optimality system, decoupling

AMS subject classifications. 49N10, 49N35, 93E20

1 Introduction

The mean-field type stochastic control problem is importance in various fields such as science, engineering, economics, management, and particularly in financial investment. The theory of mean-field forward stochastic differential equations (MF-FSDEs, for short) can be traced back to Kac [11] who presented the McKean-Vlasov stochastic differential equation motivated by a stochastic toy model for the Vlasov kinetic equation of plasma. Since then, research on related topics and their applications has become a notable and serious endeavor among researchers in applied probability and optimal stochastic controls, particularly in financial engineering. Typical representatives include, but not limited to, McKean [16], Dawson [10], Chan [7], Buckdahn–Djehiche–Li–Peng [5], Buckdahn–Li–Peng [6], Borkar–Kumar [3], Crisan–Xiong [9], Andersson–Djehiche [2], Buckdahn–Djehiche–Li [4], Meyer–Brandis–Oksendal–Zhou [17],

*Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong, China (malixun@polyu.edu.hk). This author was partially supported by Hong Kong RGC under grants 15209614, 15224215 and 15255416.

[†]Corresponding author, Department of Mathematics, National University of Singapore, 119076, Republic of Singapore (sjr@mail.ustc.edu.cn). This author was partially supported by the National Natural Science Foundation of China (11401556) and the Fundamental Research Funds for the Central Universities (WK 2040000012).

[‡]Department of Mathematics, University of Macau, Macau, China (jiexiong@umac.mo). This author acknowledges the financial support from FDCT 025/2016/A1 and MYRG2014-00015-FST.

Yong [23, 24], Sun [18], and Li–Sun–Yong [12]. The MF-FSDEs can be treated in a forward-looking way by starting with the initial state. In financial investment, however, one frequently encounters financial investment problems with future conditions (as random variables) specified. This naturally results in a *mean-field backward stochastic differential equation* (MF-BSDE, for short) with a given terminal condition (see Buckdahn–Djehiche–Li–Peng [5] and Buckdahn–Li–Peng [6]). This is an important and challenging research topic. Recently there has been increasing interest in studying this type of stochastic control problems as well as their applications. The optimal stochastic control problems under MF-BSDEs are underdeveloped in the literature, and therefore many fundamental questions remain open and methodologies need to be significantly improved.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space on which a standard one-dimensional Brownian motion $W = \{W(t); 0 \leq t < \infty\}$ is defined, where $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ is the natural filtration of W augmented by all the \mathbb{P} -null sets in \mathcal{F} . Consider the following controlled linear MF-BSDE:

$$(1.1) \quad \begin{cases} dY(s) = \left\{ A(s)Y(s) + \bar{A}(s)\mathbb{E}[Y(s)] + B(s)u(s) + \bar{B}(s)\mathbb{E}[u(s)] \right. \\ \quad \left. + C(s)Z(s) + \bar{C}(s)\mathbb{E}[Z(s)] \right\} ds + Z(s)dW(s), \quad s \in [t, T], \\ Y(T) = \xi, \end{cases}$$

where $A(\cdot), \bar{A}(\cdot), B(\cdot), \bar{B}(\cdot), C(\cdot), \bar{C}(\cdot), D(\cdot), \bar{D}(\cdot)$ are given deterministic matrix-valued functions; ξ is an \mathcal{F}_T -measurable random vector; and $u(\cdot)$ is the *control process*. The class of *admissible controls* for (1.1) is

$$\mathcal{U}[t, T] = \left\{ u : [t, T] \times \Omega \rightarrow \mathbb{R}^m \mid u(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable, } \mathbb{E} \int_t^T |u(s)|^2 ds < \infty \right\}.$$

Under some mild conditions on the coefficients of equation (1.1), for any terminal state $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$ (the set of all \mathcal{F}_T -measurable, square-integrable \mathbb{R}^n -valued processes) and any admissible control $u(\cdot) \in \mathcal{U}[t, T]$, equation (1.1) admits a unique square-integrable adapted solution $(Y(\cdot), Z(\cdot)) \equiv (Y(\cdot; \xi, u(\cdot)), Z(\cdot; \xi, u(\cdot)))$, which is called the *state process* corresponding to ξ and $u(\cdot)$. Now we introduce the following cost functional:

$$(1.2) \quad \begin{aligned} J(t, \xi; u(\cdot)) &\triangleq \mathbb{E} \left\{ \langle GY(t), Y(t) \rangle + \langle \bar{G}\mathbb{E}[Y(t)], \mathbb{E}[Y(t)] \rangle \right. \\ &\quad + \int_t^T \left[\langle Q(s)Y(s), Y(s) \rangle + \langle \bar{Q}(s)\mathbb{E}[Y(s)], \mathbb{E}[Y(s)] \rangle \right. \\ &\quad \left. + \langle R_1(s)Z(s), Z(s) \rangle + \langle \bar{R}_1(s)\mathbb{E}[Z(s)], \mathbb{E}[Z(s)] \rangle \right. \\ &\quad \left. \left. + \langle R_2(s)u(s), u(s) \rangle + \langle \bar{R}_2(s)\mathbb{E}[u(s)], \mathbb{E}[u(s)] \rangle \right] ds \right\}, \end{aligned}$$

where G, \bar{G} are symmetric matrices and $Q(\cdot), \bar{Q}(\cdot), R_i(\cdot), \bar{R}_i(\cdot)$ ($i = 1, 2$) are deterministic, symmetric matrix-valued functions. Our mean-field backward stochastic linear quadratic (LQ, for short) optimal control problem can be stated as follows.

Problem (MF-BSLQ). For any given terminal state $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$, find a $u^*(\cdot) \in \mathcal{U}[t, T]$ such that

$$(1.3) \quad J(t, \xi; u^*(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[t, T]} J(t, \xi; u(\cdot)) \triangleq V(t, \xi).$$

Any $u^*(\cdot) \in \mathcal{U}[t, T]$ satisfying (1.3) is called an *optimal control* of Problem (MF-BSLQ) for the terminal state ξ , the corresponding $(Y^*(\cdot), Z^*(\cdot)) \equiv (Y(\cdot; \xi, u^*(\cdot)), Z(\cdot; \xi, u^*(\cdot)))$ is called an *optimal state process*, and the three-tuple $(Y^*(\cdot), Z^*(\cdot), u^*(\cdot))$ is called an *optimal triple*. The function $V(\cdot, \cdot)$ is called the *value function* of Problem (MF-BSLQ). Note that when the mean-field part is absent, Problem (MF-BSLQ) is reduced to a stochastic LQ optimal control of backward stochastic differential equations (see Lim–Zhou [13] and Zhang [26] for some relevant results). For LQ optimal control problems of forward stochastic differential equations, the interested reader is referred to, for examples, [22, 8, 1, 21, 19] and the book of Yong–Zhou [25].

The rest of the paper is organized as follows. Section 2 gives some preliminaries. Among other things, we show Problem (MF-BSLQ) is uniquely solvable from a Hilbert space viewpoint. In section 3, we derive the optimality system by a variational method and the coupled two Riccati equations by a decoupling technique. Section 4 is devoted to the uniqueness and existence of solutions to the Riccati equations. In Section 5, we present explicit formulas of the optimal controls and the value function.

2 Preliminaries

Throughout this paper, $\mathbb{R}^{n \times m}$ is the Euclidean space of all $n \times m$ real matrices, \mathbb{S}^n is the space of all symmetric $n \times n$ real matrices, \mathbb{S}_+^n is the subset of \mathbb{S}^n consisting of positive definite matrices, and $\overline{\mathbb{S}_+^n}$ is the closure of \mathbb{S}_+^n in $\mathbb{R}^{n \times n}$. When $m = 1$, we simply write $\mathbb{R}^{n \times m}$ as \mathbb{R}^n , and when $n = m = 1$, we drop the superscript. Recall that the inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{n \times m}$ is given by $\langle M, N \rangle \mapsto \text{tr}(M^\top N)$, where the superscript \top denotes the transpose of matrices and $\text{tr}(K)$ denotes the trace of a matrix K , and that the induced norm on $\mathbb{R}^{n \times m}$ is given by $|M| = \sqrt{\text{tr}(M^\top M)}$. If no confusion is likely, we shall use $\langle \cdot, \cdot \rangle$ for inner products in possibly different Hilbert spaces, and denote by $|\cdot|$ the norm induced by $\langle \cdot, \cdot \rangle$. Let $t \in [0, T]$ and \mathbb{H} be a given Euclidean space. The space of \mathbb{H} -valued continuous functions on $[t, T]$ is denoted by $C([t, T]; \mathbb{H})$, and the space of \mathbb{H} -valued, p th ($1 \leq p \leq \infty$) power Lebesgue integrable functions on $[t, T]$ is denoted by $L^p(t, T; \mathbb{H})$. Further, we introduce the following spaces of random variables and stochastic processes:

$$\begin{aligned} L_{\mathcal{F}_T}^2(\Omega; \mathbb{H}) &= \left\{ \xi : \Omega \rightarrow \mathbb{H} \mid \xi \text{ is } \mathcal{F}_T\text{-measurable, } \mathbb{E}|\xi|^2 < \infty \right\}, \\ L_{\mathbb{F}}^2(t, T; \mathbb{H}) &= \left\{ \varphi : [t, T] \times \Omega \rightarrow \mathbb{H} \mid \varphi(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable,} \right. \\ &\quad \left. \mathbb{E} \int_t^T |\varphi(s)|^2 ds < \infty \right\}, \\ L_{\mathbb{F}}^2(\Omega; C([t, T]; \mathbb{H})) &= \left\{ \varphi : [t, T] \times \Omega \rightarrow \mathbb{H} \mid \varphi(\cdot) \text{ is } \mathbb{F}\text{-adapted, continuous,} \right. \\ &\quad \left. \mathbb{E} \left[\sup_{t \leq s \leq T} |\varphi(s)|^2 \right] < \infty \right\}. \end{aligned}$$

Next we introduce the following assumptions that will be in force throughout this paper.

(H1) The coefficients of the state equation satisfy the following:

$$A(\cdot), \bar{A}(\cdot) \in L^1(0, T; \mathbb{R}^{n \times n}), \quad B(\cdot), \bar{B}(\cdot) \in L^2(0, T; \mathbb{R}^{n \times m}), \quad C(\cdot), \bar{C}(\cdot) \in L^2(0, T; \mathbb{R}^{n \times n}).$$

(H2) The weighting coefficients in the cost functional satisfy

$$\begin{cases} G, \bar{G} \in \mathbb{S}^n, & Q(\cdot), \bar{Q}(\cdot) \in L^1(0, T; \mathbb{S}^n), \\ R_1(\cdot), \bar{R}_1(\cdot) \in L^\infty(0, T; \mathbb{S}^n), & R_2(\cdot), \bar{R}_2(\cdot) \in L^\infty(0, T; \mathbb{S}^m), \end{cases}$$

and there exists a constant $\delta > 0$ such that for a.e. $s \in [0, T]$,

$$\begin{cases} G, G + \bar{G} \geq 0, & Q(s), Q(s) + \bar{Q}(s) \geq 0, \\ R_1(s), R_1(s) + \bar{R}_1(s) \geq 0, & R_2(s), R_2(s) + \bar{R}_2(s) \geq \delta I. \end{cases}$$

Now we present a result concerning the well-posedness of the state equation (1.1).

Theorem 2.1. *Let (H1) hold. Then for any $(\xi, u(\cdot)) \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n) \times \mathcal{U}[t, T]$, MF-BSDE (1.1) admits a unique adapted solution $(Y(\cdot), Z(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([t, T]; \mathbb{R}^n)) \times L^2_{\mathbb{F}}(t, T; \mathbb{R}^n)$. Moreover, there exists a constant $K > 0$, independent of ξ and $u(\cdot)$, such that*

$$\mathbb{E} \left[\sup_{t \leq s \leq T} |Y(s)|^2 + \int_t^T |Z(s)|^2 ds \right] \leq K \mathbb{E} \left[|\xi|^2 + \int_t^T |u(s)|^2 ds \right].$$

Note that (H1) allows the coefficients $A(\cdot)$ and $C(\cdot)$ to be unbounded, which is a little different from the standard case [5, 6]. However, the proof of Theorem 2.1 is similar to that of the case without mean-field. We omit the proof here and refer the interested reader to Sun–Yong [20, Proposition 2.1] for details.

From Theorem 2.1, one can easily see that under (H1)–(H2), Problem (MF-BSLQ) makes sense. The following result tells us that under (H1)–(H2), Problem (MF-BSLQ) is actually uniquely solvable for any terminal state $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$.

Theorem 2.2. *Let (H1)–(H2) hold. Then for any terminal state $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$, Problem (MF-BSLQ) admits a unique optimal control.*

Proof. For any $u(\cdot) \in \mathcal{U}[t, T]$, let $(Y^u(\cdot), Z^u(\cdot))$ be the unique adapted solution to

$$(2.1) \quad \begin{cases} dY^u(s) = \left\{ A(s)Y^u(s) + \bar{A}(s)\mathbb{E}[Y^u(s)] + B(s)u(s) + \bar{B}(s)\mathbb{E}[u(s)] \right. \\ \quad \left. + C(s)Z^u(s) + \bar{C}(s)\mathbb{E}[Z^u(s)] \right\} ds + Z^u(s)dW(s), \quad s \in [t, T], \\ Y^u(T) = 0. \end{cases}$$

By the linearity of equation (2.1) and Theorem 2.1, we can define bounded linear operators $\mathcal{L} : \mathcal{U}[t, T] \rightarrow L^2_{\mathbb{F}}(\Omega; C([t, T]; \mathbb{R}^n)) \times L^2_{\mathbb{F}}(t, T; \mathbb{R}^n)$ and $\mathcal{M} : \mathcal{U}[t, T] \rightarrow L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)$ by $u(\cdot) \mapsto (Y^u(\cdot), Z^u(\cdot))$ and $u(\cdot) \mapsto Y^u(t)$, respectively, via the MF-BSDE (2.1). Similarly, we can define bounded linear operators $\mathcal{N} : L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n) \rightarrow L^2_{\mathbb{F}}(\Omega; C([t, T]; \mathbb{R}^n)) \times L^2_{\mathbb{F}}(t, T; \mathbb{R}^n)$ and $\mathcal{O} : L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n) \rightarrow L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)$ by $\xi \mapsto (Y^\xi(\cdot), Z^\xi(\cdot))$ and $\xi \mapsto Y^\xi(t)$, respectively, via the MF-BSDE

$$(2.2) \quad \begin{cases} dY^\xi(s) = \left\{ A(s)Y^\xi(s) + \bar{A}(s)\mathbb{E}[Y^\xi(s)] + C(s)Z^\xi(s) + \bar{C}(s)\mathbb{E}[Z^\xi(s)] \right\} ds \\ \quad + Z^\xi(s)dW(s), \quad s \in [t, T], \\ Y^\xi(T) = \xi. \end{cases}$$

Then for any $(\xi, u(\cdot)) \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n) \times \mathcal{U}[t, T]$, the corresponding state process $(Y(\cdot), Z(\cdot))$ and the initial value $Y(t)$ are given by

$$(Y(\cdot), Z(\cdot))^\top = \mathcal{L}u + \mathcal{N}\xi, \quad Y(t) = \mathcal{M}u + \mathcal{O}\xi.$$

Now let \mathcal{A}^* denote the adjoint of an operator \mathcal{A} , and define the bounded linear operator

$$\mathcal{Q} \triangleq \begin{pmatrix} Q + \mathbb{E}^* \bar{Q} \mathbb{E} & 0 \\ 0 & R_1 + \mathbb{E}^* \bar{R}_1 \mathbb{E} \end{pmatrix}$$

so that

$$\begin{aligned} & \mathbb{E} \int_t^T \left[\langle Q(s)Y(s), Y(s) \rangle + \langle \bar{Q}(s)\mathbb{E}[Y(s)], \mathbb{E}[Y(s)] \rangle \right. \\ & \quad \left. + \langle R_1(s)Z(s), Z(s) \rangle + \langle \bar{R}_1(s)\mathbb{E}[Z(s)], \mathbb{E}[Z(s)] \rangle \right] ds \\ &= \left\langle \begin{pmatrix} Q + \mathbb{E}^* \bar{Q} \mathbb{E} & 0 \\ 0 & R_1 + \mathbb{E}^* \bar{R}_1 \mathbb{E} \end{pmatrix} \begin{pmatrix} Y \\ Z \end{pmatrix}, \begin{pmatrix} Y \\ Z \end{pmatrix} \right\rangle \\ &= \langle \mathcal{Q}(\mathcal{L}u + \mathcal{N}\xi), \mathcal{L}u + \mathcal{N}\xi \rangle, \end{aligned}$$

where $\mathbb{E} : L^2_{\mathbb{F}}(t, T; \mathbb{H}) \rightarrow L^2(t, T; \mathbb{H})$ is defined by $\mathbb{E}[Y](s) = \mathbb{E}[Y(s)]$. Note that $\mathbb{E}^* : L^2(t, T; \mathbb{H}) \rightarrow L^2_{\mathbb{F}}(t, T; \mathbb{H})$ is the adjoint operator. Thus,

$$\begin{aligned} J(t, \xi; u(\cdot)) &= \langle (G + \mathbb{E}^* \bar{G} \mathbb{E})(\mathcal{M}u + \mathcal{O}\xi), \mathcal{M}u + \mathcal{O}\xi \rangle \\ &\quad + \langle \mathcal{Q}(\mathcal{L}u + \mathcal{N}\xi), \mathcal{L}u + \mathcal{N}\xi \rangle + \langle (R_2 + \mathbb{E}^* \bar{R}_2 \mathbb{E})u, u \rangle \\ &= \langle [\mathcal{M}^*(G + \mathbb{E}^* \bar{G} \mathbb{E})\mathcal{M} + \mathcal{L}^* \mathcal{Q} \mathcal{L} + (R_2 + \mathbb{E}^* \bar{R}_2 \mathbb{E})]u, u \rangle \\ &\quad + 2\langle [\mathcal{O}^*(G + \mathbb{E}^* \bar{G} \mathbb{E})\mathcal{M} + \mathcal{N}^* \mathcal{Q} \mathcal{L}]u, \xi \rangle \\ &\quad + \langle [\mathcal{O}^*(G + \mathbb{E}^* \bar{G} \mathbb{E})\mathcal{O} + \mathcal{N}^* \mathcal{Q} \mathcal{N}]\xi, \xi \rangle. \end{aligned}$$

Note that under (H2), we have

$$G + \mathbb{E}^* \bar{G} \mathbb{E} = G - \mathbb{E}^* G \mathbb{E} + \mathbb{E}^*(G + \bar{G})\mathbb{E} = (I - \mathbb{E}^*)G(I - \mathbb{E}) + \mathbb{E}^*(G + \bar{G})\mathbb{E} \geq 0.$$

Similarly, we can prove

$$\mathcal{Q} \geq 0, \quad R_2 + \mathbb{E}^* \bar{R}_2 \mathbb{E} \geq 0.$$

Therefore, the map $u(\cdot) \mapsto J(t, \xi; u(\cdot))$ is convex and continuous. Moreover, for any $u(\cdot) \in \mathcal{U}[t, T]$, we have by (H2):

$$\begin{aligned} \langle (R_2 + \mathbb{E}^* \bar{R}_2 \mathbb{E})u, u \rangle &= \mathbb{E} \int_t^T \left\{ \langle R_2(s)u(s), u(s) \rangle + \langle \bar{R}_2(s)\mathbb{E}[u(s)], \mathbb{E}[u(s)] \rangle \right\} ds \\ &= \frac{\delta}{2} \mathbb{E} \int_t^T |u(s)|^2 ds + \int_t^T \left\{ \mathbb{E} \langle [R_2(s) - \delta/2]u(s), u(s) \rangle - \langle [R_2(s) - \delta/2]\mathbb{E}[u(s)], \mathbb{E}[u(s)] \rangle \right\} ds \\ &\quad + \int_t^T \langle [R_2(s) + \bar{R}_2(s) - \delta/2]\mathbb{E}[u(s)], \mathbb{E}[u(s)] \rangle ds \\ &\geq \frac{\delta}{2} \mathbb{E} \int_t^T |u(s)|^2 ds. \end{aligned}$$

This further implies the map $u(\cdot) \mapsto J(t, \xi; u(\cdot))$ is strictly convex, and that

$$J(t, \xi; u(\cdot)) \rightarrow \infty \quad \text{as} \quad \mathbb{E} \int_t^T |u(s)|^2 ds \rightarrow \infty.$$

Therefore, by the basic theorem in convex analysis, for any given $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$, Problem (MF-BSLQ) has a unique optimal control. \square

3 Optimality system, decoupling, and Riccati equations

Let us first derive the optimality system for the optimal control of Problem (MF-BSLQ). For simplicity of notation, in what follows we shall often suppress the time variable s if no confusion can arise.

Theorem 3.1. *Let (H1)–(H2) hold. Let $(Y^*(\cdot), Z^*(\cdot), u^*(\cdot))$ be the optimal triple for the terminal state $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$. Then the solution $X^*(\cdot)$ to the mean-field forward stochastic differential equation (MF-FSDE, for short)*

$$(3.1) \quad \begin{cases} dX^* = \left\{ -A^\top X^* - \bar{A}^\top \mathbb{E}[X^*] + QY^* + \bar{Q}\mathbb{E}[Y^*] \right\} ds \\ \quad + \left\{ -C^\top X^* - \bar{C}^\top \mathbb{E}[X^*] + R_1 Z^* + \bar{R}_1 \mathbb{E}[Z^*] \right\} dW, \quad s \in [t, T], \\ X^*(t) = GY^*(t) + \bar{G}\mathbb{E}[Y^*(t)], \end{cases}$$

satisfies

$$(3.2) \quad R_2 u^* + \bar{R}_2 \mathbb{E}[u^*] - B^\top X^* - \bar{B}^\top \mathbb{E}[X^*] = 0, \quad \text{a.e. } s \in [t, T], \text{ a.s.}$$

Proof. For any $u(\cdot) \in \mathcal{U}[t, T]$ and any $\varepsilon \in \mathbb{R}$, let $(Y(\cdot), Z(\cdot))$ be the solution of

$$\begin{cases} dY = \{AY + \bar{A}\mathbb{E}[Y] + Bu + \bar{B}\mathbb{E}[u] + CZ + \bar{C}\mathbb{E}[Z]\} ds + Z dW, \quad s \in [t, T], \\ Y(T) = 0, \end{cases}$$

and let $(Y^\varepsilon(\cdot), Z^\varepsilon(\cdot))$ be the solution to the perturbed state equation

$$\begin{cases} dY^\varepsilon = \{AY^\varepsilon + \bar{A}\mathbb{E}[Y^\varepsilon] + B(u^* + \varepsilon u) + \bar{B}\mathbb{E}[u^* + \varepsilon u] + CZ^\varepsilon + \bar{C}\mathbb{E}[Z^\varepsilon]\} ds + Z^\varepsilon dW, \\ Y^\varepsilon(T) = \xi. \end{cases}$$

It is clear that $(Y^\varepsilon(\cdot), Z^\varepsilon(\cdot)) = (Y^*(\cdot) + \varepsilon Y(\cdot), Z^*(\cdot) + \varepsilon Z(\cdot))$, and hence

$$\begin{aligned} & J(t, \xi; u^*(\cdot) + \varepsilon u(\cdot)) - J(t, \xi; u^*(\cdot)) \\ &= 2\varepsilon \mathbb{E} \left\{ \langle GY^*(t), Y(t) \rangle + \langle \bar{G}\mathbb{E}[Y^*(t)], \mathbb{E}[Y(t)] \rangle + \int_t^T \left(\langle QY^*, Y \rangle + \langle R_1 Z^*, Z \rangle + \langle R_2 u^*, u \rangle \right) ds \right. \\ & \quad \left. + \int_t^T \left(\langle \bar{Q}\mathbb{E}[Y^*], \mathbb{E}[Y] \rangle + \langle \bar{R}_1 \mathbb{E}[Z^*], \mathbb{E}[Z] \rangle + \langle \bar{R}_2 \mathbb{E}[u^*], \mathbb{E}[u] \rangle \right) ds \right\} \\ & \quad + \varepsilon^2 \mathbb{E} \left\{ \langle GY(t), Y(t) \rangle + \langle \bar{G}\mathbb{E}[Y(t)], \mathbb{E}[Y(t)] \rangle + \int_t^T \left(\langle QY, Y \rangle + \langle R_1 Z, Z \rangle + \langle R_2 u, u \rangle \right) ds \right. \\ & \quad \left. + \int_t^T \left(\langle \bar{Q}\mathbb{E}[Y], \mathbb{E}[Y] \rangle + \langle \bar{R}_1 \mathbb{E}[Z], \mathbb{E}[Z] \rangle + \langle \bar{R}_2 \mathbb{E}[u], \mathbb{E}[u] \rangle \right) ds \right\}. \end{aligned}$$

Applying Itô's formula to $s \mapsto \langle X^*(s), Y(s) \rangle$, we have

$$\begin{aligned} & -\mathbb{E} \left\{ \langle GY^*(t), Y(t) \rangle + \langle \bar{G}\mathbb{E}[Y^*(t)], \mathbb{E}[Y(t)] \rangle \right\} = -\mathbb{E} \langle GY^*(t) + \bar{G}\mathbb{E}[Y^*(t)], Y(t) \rangle \\ &= \mathbb{E} \int_t^T \left\{ \langle QY^* + \bar{Q}\mathbb{E}[Y^*], Y \rangle + \langle R_1 Z^* + \bar{R}_1 \mathbb{E}[Z^*], Z \rangle + \langle B^\top X^* + \bar{B}^\top \mathbb{E}[X^*], u \rangle \right\} ds. \end{aligned}$$

It follows that for any $u(\cdot) \in \mathcal{U}[t, T]$ and any $\varepsilon \in \mathbb{R}$,

$$J(t, \xi; u^*(\cdot) + \varepsilon u(\cdot)) - J(t, \xi; u^*(\cdot))$$

$$\begin{aligned}
&= 2\varepsilon \mathbb{E} \int_t^T \langle R_2 u^* + \bar{R}_2 \mathbb{E}[u^*] - B^\top X^* - \bar{B}^\top \mathbb{E}[X^*], u \rangle ds \\
&\quad + \varepsilon^2 \mathbb{E} \left\{ \langle GY(t), Y(t) \rangle + \langle \bar{G} \mathbb{E}[Y(t)], \mathbb{E}[Y(t)] \rangle + \int_t^T \left(\langle QY, Y \rangle + \langle R_1 Z, Z \rangle + \langle R_2 u, u \rangle \right) ds \right. \\
&\quad \left. + \int_t^T \left(\langle \bar{Q} \mathbb{E}[Y], \mathbb{E}[Y] \rangle + \langle \bar{R}_1 \mathbb{E}[Z], \mathbb{E}[Z] \rangle + \langle \bar{R}_2 \mathbb{E}[u], \mathbb{E}[u] \rangle \right) ds \right\}.
\end{aligned}$$

Since $u^*(\cdot)$ is the optimal control of Problem (MF-BSLQ) for the terminal state ξ , dividing by ε in the above and then letting $\varepsilon \rightarrow 0$, we obtain

$$\mathbb{E} \int_t^T \langle R_2 u^* + \bar{R}_2 \mathbb{E}[u^*] - B^\top X^* - \bar{B}^\top \mathbb{E}[X^*], u \rangle ds = 0, \quad \forall u(\cdot) \in \mathcal{U}[t, T],$$

from which (3.2) follows immediately. \square

From the above result, we see that if $u(\cdot)$ happens to be an optimal control of Problem (MF-BSLQ) for terminal state ξ , then the following mean-field forward-backward stochastic differential equation (MF-FBSDE, for short) admits an adapted solution $(X(\cdot), Y(\cdot), Z(\cdot))$:

$$(3.3) \quad \begin{cases} dX = \left\{ -A^\top X - \bar{A}^\top \mathbb{E}[X] + QY + \bar{Q} \mathbb{E}[Y] \right\} ds \\ \quad + \left\{ -C^\top X - \bar{C}^\top \mathbb{E}[X] + R_1 Z + \bar{R}_1 \mathbb{E}[Z] \right\} dW, \quad s \in [t, T], \\ dY = \left\{ AY + \bar{A} \mathbb{E}[Y] + Bu + \bar{B} \mathbb{E}[u] + CZ + \bar{C} \mathbb{E}[Z] \right\} ds + Z dW, \quad s \in [t, T], \\ X(t) = GY(t) + \bar{G} \mathbb{E}[Y(t)], \quad Y(T) = \xi, \end{cases}$$

and the following *stationarity condition* holds:

$$(3.4) \quad R_2 u + \bar{R}_2 \mathbb{E}[u] - B^\top X - \bar{B}^\top \mathbb{E}[X] = 0, \quad \text{a.e. } s \in [t, T], \text{ a.s.}$$

We call (3.3), together with the stationarity condition (3.4), the *optimality system* for the optimal control of Problem (MF-BSLQ). Note that (3.4) brings a coupling into the MF-FBSDE (3.3) and does not provide a representation for $u(\cdot)$ because the equation for $X(\cdot)$ involves $Y(\cdot)$ and $Z(\cdot)$.

To solve the optimality system (3.3)–(3.4), we use the decoupling technique inspired by the four-step scheme introduced in [14, 15] for general FBSDEs. This will lead to a derivation of two Riccati equations. To be precise, we conjecture that $X(\cdot)$ and $Y(\cdot)$ are related by the following:

$$(3.5) \quad Y(s) = -\Sigma(s) \{ X(s) - \mathbb{E}[X(s)] \} - \Gamma(s) \mathbb{E}[X(s)] - \varphi(s), \quad s \in [t, T],$$

where $\Sigma(\cdot), \Gamma(\cdot) : [0, T] \rightarrow \mathbb{S}^n$ are absolutely continuous and $\varphi(\cdot)$ satisfies

$$(3.6) \quad d\varphi(s) = \alpha(s) ds + \beta(s) dW(s), \quad \varphi(T) = -\xi,$$

for some \mathbb{F} -progressively measurable processes $\alpha(\cdot)$ and $\beta(\cdot)$. Note that

$$(3.7) \quad \begin{cases} d\mathbb{E}[X] = \left\{ -(A + \bar{A})^\top \mathbb{E}[X] + (Q + \bar{Q}) \mathbb{E}[Y] \right\} ds \\ d\mathbb{E}[Y] = \left\{ (A + \bar{A}) \mathbb{E}[Y] + (B + \bar{B}) \mathbb{E}[u] + (C + \bar{C}) \mathbb{E}[Z] \right\} ds, \\ \mathbb{E}[X(t)] = (G + \bar{G}) \mathbb{E}[Y(t)], \quad \mathbb{E}[Y(T)] = \mathbb{E}[\xi], \\ (R_2 + \bar{R}_2) \mathbb{E}[u] - (B + \bar{B})^\top \mathbb{E}[X] = 0. \end{cases}$$

Thus,

$$(3.8) \quad \begin{cases} d(X - \mathbb{E}[X]) = \left\{ -A^\top (X - \mathbb{E}[X]) + Q(Y - \mathbb{E}[Y]) \right\} ds \\ \quad + \left\{ -C^\top X - \bar{C}^\top \mathbb{E}[X] + R_1 Z + \bar{R}_1 \mathbb{E}[Z] \right\} dW, \\ d(Y - \mathbb{E}[Y]) = \left\{ A(Y - \mathbb{E}[Y]) + B(u - \mathbb{E}[u]) + C(Z - \mathbb{E}[Z]) \right\} ds + Z dW, \\ X(t) - \mathbb{E}[X(t)] = G(Y(t) - \mathbb{E}[Y(t)]), \quad Y(T) - \mathbb{E}[Y(T)] = \xi - \mathbb{E}[\xi], \\ R_2(u - \mathbb{E}[u]) - B^\top (X - \mathbb{E}[X]) = 0. \end{cases}$$

From (3.5) we have

$$(3.9) \quad Y - \mathbb{E}[Y] = -\Sigma(X - \mathbb{E}[X]) - (\varphi - \mathbb{E}[\varphi]), \quad \mathbb{E}[Y] = -\Gamma \mathbb{E}[X] - \mathbb{E}[\varphi].$$

Denoting $\eta(\cdot) = \varphi(\cdot) - \mathbb{E}[\varphi(\cdot)]$ and $\gamma(\cdot) = \alpha(\cdot) - \mathbb{E}[\alpha(\cdot)]$, we have from (3.6) that

$$(3.10) \quad d\eta(s) = \gamma(s)ds + \beta(s)dW(s), \quad \eta(T) = \mathbb{E}[\xi] - \xi.$$

Then (3.8)–(3.10) yield

$$\begin{aligned} 0 &= d(Y - \mathbb{E}[Y]) + \dot{\Sigma}(X - \mathbb{E}[X])ds + \Sigma d(X - \mathbb{E}[X]) + d\eta \\ &= \left\{ A(Y - \mathbb{E}[Y]) + B(u - \mathbb{E}[u]) + C(Z - \mathbb{E}[Z]) \right\} ds + Z dW \\ &\quad + \dot{\Sigma}(X - \mathbb{E}[X])ds + \left\{ -\Sigma A^\top (X - \mathbb{E}[X]) + \Sigma Q(Y - \mathbb{E}[Y]) \right\} ds \\ &\quad + \left\{ -\Sigma C^\top X - \Sigma \bar{C}^\top \mathbb{E}[X] + \Sigma R_1 Z + \Sigma \bar{R}_1 \mathbb{E}[Z] \right\} dW + \gamma ds + \beta dW \\ &= \left\{ A(Y - \mathbb{E}[Y]) + B(u - \mathbb{E}[u]) + C(Z - \mathbb{E}[Z]) + \dot{\Sigma}(X - \mathbb{E}[X]) \right. \\ &\quad \left. - \Sigma A^\top (X - \mathbb{E}[X]) + \Sigma Q(Y - \mathbb{E}[Y]) + \gamma \right\} ds \\ &\quad + \left\{ Z - \Sigma C^\top X - \Sigma \bar{C}^\top \mathbb{E}[X] + \Sigma R_1 Z + \Sigma \bar{R}_1 \mathbb{E}[Z] + \beta \right\} dW \\ &= \left\{ -A\Sigma(X - \mathbb{E}[X]) - A\eta + BR_2^{-1}B^\top (X - \mathbb{E}[X]) + C(Z - \mathbb{E}[Z]) \right. \\ &\quad \left. + \dot{\Sigma}(X - \mathbb{E}[X]) - \Sigma A^\top (X - \mathbb{E}[X]) - \Sigma Q\Sigma(X - \mathbb{E}[X]) - \Sigma Q\eta + \gamma \right\} ds \\ &\quad + \left\{ Z - \Sigma C^\top X - \Sigma \bar{C}^\top \mathbb{E}[X] + \Sigma R_1 Z + \Sigma \bar{R}_1 \mathbb{E}[Z] + \beta \right\} dW \\ &= \left\{ (\dot{\Sigma} - A\Sigma - \Sigma A^\top - \Sigma Q\Sigma + BR_2^{-1}B^\top)(X - \mathbb{E}[X]) \right. \\ &\quad \left. + C(Z - \mathbb{E}[Z]) - (A + \Sigma Q)\eta + \gamma \right\} ds \\ &\quad + \left\{ Z - \Sigma C^\top X - \Sigma \bar{C}^\top \mathbb{E}[X] + \Sigma R_1 Z + \Sigma \bar{R}_1 \mathbb{E}[Z] + \beta \right\} dW. \end{aligned}$$

This implies

$$(3.11) \quad (\dot{\Sigma} - A\Sigma - \Sigma A^\top - \Sigma Q\Sigma + BR_2^{-1}B^\top)(X - \mathbb{E}[X]) + C(Z - \mathbb{E}[Z]) - (A + \Sigma Q)\eta + \gamma = 0,$$

$$(3.12) \quad Z - \Sigma C^\top X - \Sigma \bar{C}^\top \mathbb{E}[X] + \Sigma R_1 Z + \Sigma \bar{R}_1 \mathbb{E}[Z] + \beta = 0.$$

Now from (3.12) we have

$$(3.13) \quad (I + \Sigma R_1 + \Sigma \bar{R}_1)\mathbb{E}[Z] - \Sigma(C + \bar{C})^\top \mathbb{E}[X] + \mathbb{E}[\beta] = 0.$$

Subtracting (3.13) from (3.12), we obtain

$$(3.14) \quad (I + \Sigma R_1)(Z - \mathbb{E}[Z]) - \Sigma C^\top (X - \mathbb{E}[X]) + (\beta - \mathbb{E}[\beta]) = 0.$$

Assuming that $I + \Sigma R_1$ and $I + \Sigma R_1 + \Sigma \bar{R}_1$ are invertible, we obtain from (3.13) and (3.14):

$$(3.15) \quad \mathbb{E}[Z] = (I + \Sigma R_1 + \Sigma \bar{R}_1)^{-1} \{ \Sigma(C + \bar{C})^\top \mathbb{E}[X] - \mathbb{E}[\beta] \},$$

$$(3.16) \quad Z - \mathbb{E}[Z] = (I + \Sigma R_1)^{-1} \{ \Sigma C^\top (X - \mathbb{E}[X]) - (\beta - \mathbb{E}[\beta]) \}.$$

Substitution of (3.16) into (3.11) now gives

$$\begin{aligned} & \left[\dot{\Sigma} - A\Sigma - \Sigma A^\top - \Sigma Q \Sigma + B R_2^{-1} B^\top + C(I + \Sigma R_1)^{-1} \Sigma C^\top \right] (X - \mathbb{E}[X]) \\ & - C(I + \Sigma R_1)^{-1} (\beta - \mathbb{E}[\beta]) - (A + \Sigma Q)\eta + \gamma = 0, \end{aligned}$$

from which one should let

$$(3.17) \quad \begin{cases} \dot{\Sigma} - A\Sigma - \Sigma A^\top - \Sigma Q \Sigma + B R_2^{-1} B^\top + C(I + \Sigma R_1)^{-1} \Sigma C^\top = 0, \\ \gamma - C(I + \Sigma R_1)^{-1} (\beta - \mathbb{E}[\beta]) - (A + \Sigma Q)\eta = 0. \end{cases}$$

Also, we have from (3.7), (3.9), and (3.15):

$$\begin{aligned} 0 &= \frac{d}{ds} (\mathbb{E}[Y] + \Gamma \mathbb{E}[X] + \mathbb{E}[\varphi]) \\ &= (A + \bar{A}) \mathbb{E}[Y] + (B + \bar{B}) \mathbb{E}[u] + (C + \bar{C}) \mathbb{E}[Z] \\ &\quad + \dot{\Gamma} \mathbb{E}[X] - \Gamma(A + \bar{A})^\top \mathbb{E}[X] + \Gamma(Q + \bar{Q}) \mathbb{E}[Y] + \mathbb{E}[\alpha] \\ &= - (A + \bar{A}) \Gamma \mathbb{E}[X] - (A + \bar{A}) \mathbb{E}[\varphi] + (B + \bar{B}) (R_2 + \bar{R}_2)^{-1} (B + \bar{B})^\top \mathbb{E}[X] \\ &\quad + (C + \bar{C}) (I + \Sigma R_1 + \Sigma \bar{R}_1)^{-1} \{ \Sigma(C + \bar{C})^\top \mathbb{E}[X] - \mathbb{E}[\beta] \} \\ &\quad + \dot{\Gamma} \mathbb{E}[X] - \Gamma(A + \bar{A})^\top \mathbb{E}[X] - \Gamma(Q + \bar{Q}) \Gamma \mathbb{E}[X] - \Gamma(Q + \bar{Q}) \mathbb{E}[\varphi] + \mathbb{E}[\alpha] \\ &= \{ \dot{\Gamma} - (A + \bar{A}) \Gamma - \Gamma(A + \bar{A})^\top - \Gamma(Q + \bar{Q}) \Gamma + (B + \bar{B}) (R_2 + \bar{R}_2)^{-1} (B + \bar{B})^\top \\ &\quad + (C + \bar{C}) (I + \Sigma R_1 + \Sigma \bar{R}_1)^{-1} \Sigma(C + \bar{C})^\top \} \mathbb{E}[X] \\ &\quad - [(A + \bar{A}) + \Gamma(Q + \bar{Q})] \mathbb{E}[\varphi] - (C + \bar{C}) (I + \Sigma R_1 + \Sigma \bar{R}_1)^{-1} \mathbb{E}[\beta] + \mathbb{E}[\alpha]. \end{aligned}$$

Hence, one should let

$$(3.18) \quad \begin{cases} \dot{\Gamma} - (A + \bar{A}) \Gamma - \Gamma(A + \bar{A})^\top - \Gamma(Q + \bar{Q}) \Gamma + (B + \bar{B}) (R_2 + \bar{R}_2)^{-1} (B + \bar{B})^\top \\ \quad + (C + \bar{C}) (I + \Sigma R_1 + \Sigma \bar{R}_1)^{-1} \Sigma(C + \bar{C})^\top = 0, \\ \mathbb{E}[\alpha] - [(A + \bar{A}) + \Gamma(Q + \bar{Q})] \mathbb{E}[\varphi] - (C + \bar{C}) (I + \Sigma R_1 + \Sigma \bar{R}_1)^{-1} \mathbb{E}[\beta] = 0. \end{cases}$$

Moreover, comparing the terminal values on both sides of the two equations in (3.9), one has

$$\Sigma(T) = 0, \quad \Gamma(T) = 0.$$

Therefore, by (3.17)–(3.18), we see that $\Sigma(\cdot)$ and $\Gamma(\cdot)$ should satisfy the following Riccati-type equations, respectively:

$$(3.19) \quad \begin{cases} \dot{\Sigma} - A\Sigma - \Sigma A^\top - \Sigma Q \Sigma + B R_2^{-1} B^\top + C(I + \Sigma R_1)^{-1} \Sigma C^\top = 0, & s \in [0, T], \\ \Sigma(T) = 0, \end{cases}$$

$$(3.20) \quad \begin{cases} \dot{\Gamma} - (A + \bar{A}) \Gamma - \Gamma(A + \bar{A})^\top - \Gamma(Q + \bar{Q}) \Gamma + (B + \bar{B}) (R_2 + \bar{R}_2)^{-1} (B + \bar{B})^\top \\ \quad + (C + \bar{C}) (I + \Sigma R_1 + \Sigma \bar{R}_1)^{-1} \Sigma(C + \bar{C})^\top = 0, & s \in [0, T], \\ \Gamma(T) = 0, \end{cases}$$

and $\varphi(\cdot)$ should satisfy the following MF-BSDE on $[0, T]$:

$$(3.21) \quad \begin{cases} d\varphi = \left\{ (A + \Sigma Q)\varphi + [\bar{A} + \Gamma(Q + \bar{Q}) - \Sigma Q]\mathbb{E}[\varphi] + C(I + \Sigma R_1)^{-1}\beta \right. \\ \quad \left. + [(C + \bar{C})(I + \Sigma R_1 + \Sigma \bar{R}_1)^{-1} - C(I + \Sigma R_1)^{-1}]\mathbb{E}[\beta] \right\} ds + \beta dW, \\ \varphi(T) = -\xi. \end{cases}$$

4 Unique solvability of Riccati equations

In this section we shall establish the unique solvability of the Riccati equations (3.19) and (3.20). Once $\Sigma(\cdot)$ and $\Pi(\cdot)$ are known, the existence of a solution to MF-BSDE (3.21) will immediately follow from Theorem 2.1.

Theorem 4.1. *Let (H1)–(H2) hold. Then the Riccati equations (3.19) and (3.20) admit unique solutions $\Sigma(\cdot) \in C([0, T]; \overline{\mathbb{S}}_+^n)$ and $\Gamma(\cdot) \in C([0, T]; \overline{\mathbb{S}}_+^n)$, respectively.*

Proof. For $\lambda > 0$ and $\varepsilon \geq 0$, let us consider the forward stochastic differential equation (FSDE, for short)

$$\begin{cases} dX(s) = \left\{ A(s)X(s) + \bar{A}(s)\mathbb{E}[X(s)] + B(s)u(s) + \bar{B}(s)\mathbb{E}[u(s)] \right. \\ \quad \left. + C(s)v(s) + \bar{C}(s)\mathbb{E}[v(s)] \right\} ds + v(s)dW(s), \quad s \in [t, T], \\ X(t) = \xi, \end{cases}$$

and the cost functional

$$\begin{aligned} J_{\lambda, \varepsilon}(t, \xi; u(\cdot), v(\cdot)) \\ \triangleq \mathbb{E} \left\{ \int_t^T \left[\langle Q(s)X(s), X(s) \rangle + \langle \bar{Q}(s)\mathbb{E}[X(s)], \mathbb{E}[X(s)] \rangle \right. \right. \\ \quad + \langle [\varepsilon I + R_1(s)]v(s), v(s) \rangle + \langle \bar{R}_1(s)\mathbb{E}[v(s)], \mathbb{E}[v(s)] \rangle \\ \quad \left. \left. + \langle R_2(s)u(s), u(s) \rangle + \langle \bar{R}_2(s)\mathbb{E}[u(s)], \mathbb{E}[u(s)] \rangle \right] ds + \lambda |X(T)|^2 \right\}. \end{aligned}$$

We pose the following forward mean-field LQ problem: For any given initial pair $(t, \xi) \in [0, T] \times L_{\mathcal{F}_t}^2(\Omega; \mathbb{R}^n)$, find a pair $(u^*(\cdot), v^*(\cdot)) \in L_{\mathbb{F}}^2(t, T; \mathbb{R}^m) \times L_{\mathbb{F}}^2(t, T; \mathbb{R}^n)$ such that

$$J_{\lambda, \varepsilon}(t, \xi; u^*(\cdot), v^*(\cdot)) = \inf_{u(\cdot), v(\cdot)} J_{\lambda, \varepsilon}(t, \xi; u(\cdot), v(\cdot)) \triangleq V_{\lambda, \varepsilon}(t, \xi)$$

as $(u(\cdot), v(\cdot))$ ranges over the space $L_{\mathbb{F}}^2(t, T; \mathbb{R}^m) \times L_{\mathbb{F}}^2(t, T; \mathbb{R}^n)$. By (H2), we have for any $(t, \xi) \in [0, T] \times L_{\mathcal{F}_t}^2(\Omega; \mathbb{R}^n)$ and any $(u(\cdot), v(\cdot)) \in L_{\mathbb{F}}^2(t, T; \mathbb{R}^m) \times L_{\mathbb{F}}^2(t, T; \mathbb{R}^n)$,

$$\begin{aligned} J_{\lambda, \varepsilon}(t, \xi; u(\cdot), v(\cdot)) \\ \geq \int_t^T \left\{ \langle Q(s)\mathbb{E}[X(s)], \mathbb{E}[X(s)] \rangle + \langle \bar{Q}(s)\mathbb{E}[X(s)], \mathbb{E}[X(s)] \rangle \right. \\ \quad + \langle R_1(s)\mathbb{E}[v(s)], \mathbb{E}[v(s)] \rangle + \langle \bar{R}_1(s)\mathbb{E}[v(s)], \mathbb{E}[v(s)] \rangle \\ \quad \left. + \langle [R_2(s) - \delta/2]\mathbb{E}[u(s)], \mathbb{E}[u(s)] \rangle + \langle \bar{R}_2(s)\mathbb{E}[u(s)], \mathbb{E}[u(s)] \rangle \right\} ds \\ + \varepsilon \mathbb{E} \int_t^T |v(s)|^2 ds + \frac{\delta}{2} \mathbb{E} \int_t^T |u(s)|^2 ds \\ \geq \left(\varepsilon \wedge \frac{\delta}{2} \right) \mathbb{E} \int_t^T \left[|v(s)|^2 + |u(s)|^2 \right] ds. \end{aligned} \tag{4.1}$$

Then it follows from [18, Theorem 5.2] (see also [23, Theorem 4.1]) that for any $\lambda, \varepsilon > 0$, the following two Riccati equations

$$(4.2) \quad \begin{cases} \dot{P}_{\lambda,\varepsilon} + P_{\lambda,\varepsilon}A + A^\top P_{\lambda,\varepsilon} + Q - P_{\lambda,\varepsilon}(B, C) \begin{pmatrix} R_2 & 0 \\ 0 & \varepsilon I + R_1 + P_{\lambda,\varepsilon} \end{pmatrix}^{-1} (B, C)^\top P_{\lambda,\varepsilon} = 0, \\ P_{\lambda,\varepsilon}(T) = \lambda I, \end{cases}$$

and

$$(4.3) \quad \begin{cases} \dot{\Pi}_{\lambda,\varepsilon} + \Pi_{\lambda,\varepsilon}(A + \bar{A}) + (A + \bar{A})^\top \Pi_{\lambda,\varepsilon} + Q + \bar{Q} \\ - \Pi_{\lambda,\varepsilon}(B + \bar{B}, C + \bar{C}) \begin{pmatrix} R_2 + \bar{R}_2 & 0 \\ 0 & \varepsilon I + R_1 + \bar{R}_1 + P_{\lambda,\varepsilon} \end{pmatrix}^{-1} (B + \bar{B}, C + \bar{C})^\top \Pi_{\lambda,\varepsilon} = 0, \\ \Pi_{\lambda,\varepsilon}(T) = \lambda I, \end{cases}$$

admit unique solutions $P_{\lambda,\varepsilon}(\cdot)$ and $\Pi_{\lambda,\varepsilon}(\cdot)$, respectively, such that

$$(4.4) \quad \begin{aligned} V_{\lambda,\varepsilon}(t, \xi) &= \mathbb{E} \langle P_{\lambda,\varepsilon}(t)(\xi - \mathbb{E}[\xi]), \xi - \mathbb{E}[\xi] \rangle + \langle \Pi_{\lambda,\varepsilon}(t) \mathbb{E}[\xi], \mathbb{E}[\xi] \rangle, \\ \forall (t, \xi) &\in [0, T] \times L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^n). \end{aligned}$$

For fixed $\lambda > 0$, we have

$$(4.5) \quad \begin{aligned} V_{\lambda,\varepsilon}(t, \xi) &= \inf_{u(\cdot), v(\cdot)} J_{\lambda,\varepsilon}(t, \xi; u(\cdot), v(\cdot)) \leq \inf_{u(\cdot), v(\cdot)} J_{\lambda,\varepsilon'}(t, \xi; u(\cdot), v(\cdot)) = V_{\lambda,\varepsilon'}(t, \xi), \\ \forall (t, \xi) &\in [0, T] \times L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^n), \end{aligned}$$

whenever $0 \leq \varepsilon \leq \varepsilon'$. This implies

$$(4.6) \quad P_{\lambda,\varepsilon}(t) \leq P_{\lambda,\varepsilon'}(t), \quad \Pi_{\lambda,\varepsilon}(t) \leq \Pi_{\lambda,\varepsilon'}(t), \quad \forall t \in [0, T]; \quad \forall 0 < \varepsilon \leq \varepsilon'.$$

On the other hand, we may conclude from (4.1) that

$$V_{\lambda,0}(t, \xi) > 0, \quad \forall (t, \xi) \in [0, T] \times L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^n) \text{ with } \xi \neq 0,$$

which, together with (4.5) and (4.6), implies that the limits $\lim_{\varepsilon \rightarrow 0} P_{\lambda,\varepsilon}(t)$ and $\lim_{\varepsilon \rightarrow 0} \Pi_{\lambda,\varepsilon}(t)$ exist, and

$$P_\lambda(t) \triangleq \lim_{\varepsilon \rightarrow 0} P_{\lambda,\varepsilon}(t) > 0, \quad \Pi_\lambda(t) \triangleq \lim_{\varepsilon \rightarrow 0} \Pi_{\lambda,\varepsilon}(t) > 0, \quad \forall t \in [0, T].$$

By (4.2), we get

$$\begin{aligned} P_{\lambda,\varepsilon}(t) &= \lambda I + \int_t^T \left[P_{\lambda,\varepsilon}A + A^\top P_{\lambda,\varepsilon} + Q \right. \\ &\quad \left. - P_{\lambda,\varepsilon}(B, C) \begin{pmatrix} R_2 & 0 \\ 0 & \varepsilon I + R_1 + P_{\lambda,\varepsilon} \end{pmatrix}^{-1} (B, C)^\top P_{\lambda,\varepsilon} \right] ds. \end{aligned}$$

Passing to limit as $\varepsilon \rightarrow 0$, by the bounded convergence theorem, we have

$$P_\lambda(t) = \lambda I + \int_t^T \left[P_\lambda A + A^\top P_\lambda + Q - P_\lambda(B, C) \begin{pmatrix} R_2 & 0 \\ 0 & R_1 + P_\lambda \end{pmatrix}^{-1} (B, C)^\top P_\lambda \right] ds.$$

Therefore,

$$\begin{cases} \dot{P}_\lambda + P_\lambda A + A^\top P_\lambda + Q - P_\lambda(B, C) \begin{pmatrix} R_2 & 0 \\ 0 & R_1 + P_\lambda \end{pmatrix}^{-1} (B, C)^\top P_\lambda = 0, \\ P_\lambda(T) = \lambda I. \end{cases}$$

Similarly using (4.3), we have

$$\begin{cases} \dot{\Pi}_\lambda + \Pi_\lambda(A + \bar{A}) + (A + \bar{A})^\top \Pi_\lambda + Q + \bar{Q} \\ - \Pi_\lambda(B + \bar{B}, C + \bar{C}) \begin{pmatrix} R_2 + \bar{R}_2 & 0 \\ 0 & R_1 + \bar{R}_1 + P_\lambda \end{pmatrix}^{-1} (B + \bar{B}, C + \bar{C})^\top \Pi_\lambda = 0, \\ \Pi_\lambda(T) = \lambda I. \end{cases}$$

Next, for fixed $\varepsilon > 0$, we have

$$V_{\lambda, \varepsilon}(t, \xi) = \inf_{u(\cdot), v(\cdot)} J_{\lambda, \varepsilon}(t, \xi; u(\cdot), v(\cdot)) \leq \inf_{u(\cdot), v(\cdot)} J_{\lambda', \varepsilon}(t, \xi; u(\cdot), v(\cdot)) = V_{\lambda', \varepsilon}(t, \xi), \\ \forall (t, \xi) \in [0, T] \times L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^n),$$

whenever $0 < \lambda \leq \lambda'$. It follows that

$$P_{\lambda, \varepsilon}(t) \leq P_{\lambda', \varepsilon}(t), \quad \Pi_{\lambda, \varepsilon}(t) \leq \Pi_{\lambda', \varepsilon}(t), \quad \forall t \in [0, T],$$

and hence

$$0 < P_\lambda(t) \leq P_{\lambda'}(t), \quad 0 < \Pi_\lambda(t) \leq \Pi_{\lambda'}(t), \quad \forall t \in [0, T]; \quad 0 < \lambda \leq \lambda'.$$

Therefore, the families $\{\Sigma_\lambda(t) \triangleq P_\lambda(t)^{-1} : \lambda > 0\}$ and $\{\Gamma_\lambda(t) \triangleq \Pi_\lambda(t)^{-1} : \lambda > 0\}$ are decreasing in \mathbb{S}_+^n and hence converge. We denote

$$\Sigma(t) = \lim_{\lambda \rightarrow \infty} \Sigma_\lambda(t) \geq 0, \quad \Gamma(t) = \lim_{\lambda \rightarrow \infty} \Gamma_\lambda(t) \geq 0, \quad t \in [0, T].$$

Now using the fact

$$\begin{cases} \frac{d}{dt} [P_\lambda(t)^{-1} P_\lambda(t)] = 0, & \frac{d}{dt} [\Pi_\lambda(t)^{-1} \Pi_\lambda(t)] = 0, \\ [R_1(t) + P_\lambda(t)]^{-1} = [I + P_\lambda(t)^{-1} R_1(t)]^{-1} P_\lambda(t)^{-1}, \\ [R_1(t) + \bar{R}_1(t) + P_\lambda(t)]^{-1} = \{I + P_\lambda(t)^{-1} [R_1(t) + \bar{R}_1(t)]\}^{-1} P_\lambda(t)^{-1}, \end{cases}$$

one can easily show that $\Sigma_\lambda(\cdot)$ is a solution of

$$(4.7) \quad \begin{cases} \dot{\Sigma}_\lambda - A \Sigma_\lambda - \Sigma_\lambda A^\top - \Sigma_\lambda Q \Sigma_\lambda + B R_2^{-1} B^\top + C(I + \Sigma_\lambda R_1)^{-1} \Sigma_\lambda C^\top = 0, \\ \Sigma_\lambda(T) = \lambda^{-1} I, \end{cases}$$

and $\Gamma_\lambda(\cdot)$ is a solution of

$$(4.8) \quad \begin{cases} \dot{\Gamma}_\lambda - (A + \bar{A}) \Gamma_\lambda - \Gamma_\lambda (A + \bar{A})^\top - \Gamma_\lambda (Q + \bar{Q}) \Gamma_\lambda + (B + \bar{B})(R_2 + \bar{R}_2)^{-1} (B + \bar{B})^\top \\ + (C + \bar{C}) [I + \Sigma_\lambda (R_1 + \bar{R}_1)]^{-1} \Sigma_\lambda (C + \bar{C})^\top = 0, \\ \Gamma_\lambda(T) = \lambda^{-1} I. \end{cases}$$

Note that (4.7) is equivalent to

$$\Sigma_\lambda(t) = \lambda^{-1}I - \int_t^T \left[A\Sigma_\lambda + \Sigma_\lambda A^\top + \Sigma_\lambda Q \Sigma_\lambda - BR_2^{-1}B^\top - C(I + \Sigma_\lambda R_1)^{-1}\Sigma_\lambda C^\top \right] ds.$$

Because $\{\Sigma_\lambda(t)\}_{\lambda \geq 1}$ and $\{[I + \Sigma_\lambda(t)R_1(t)]^{-1}\Sigma_\lambda(t)\}_{\lambda \geq 1}$ are uniformly bounded on $[0, T]$, by letting $\lambda \rightarrow \infty$, we obtain from the dominated convergence theorem:

$$\Sigma(t) = - \int_t^T \left[A\Sigma + \Sigma A^\top + \Sigma Q \Sigma - BR_2^{-1}B^\top - C(I + \Sigma R_1)^{-1}\Sigma C^\top \right] ds,$$

so $\Sigma(\cdot)$ is a solution of the Riccati equation (3.19). Likewise, $\Gamma(\cdot)$ is a solution of the Riccati equation (3.20).

To prove the uniqueness, let us suppose that $\Sigma_1(\cdot), \Sigma_2(\cdot) \in C([0, T]; \overline{\mathbb{S}}_+^n)$ are two solutions of (3.19). Then it is easy to show that $\Delta(\cdot) \triangleq \Sigma_1(\cdot) - \Sigma_2(\cdot)$ is a solution to the equation

$$\begin{cases} \dot{\Delta} - (A + \Sigma_1 Q)\Delta - \Delta(A + \Sigma_2 Q)^\top + C(I + \Sigma_1 R_1)^{-1}\Delta[I - R_1(I + \Sigma_2 R_1)^{-1}\Sigma_2]C^\top = 0, \\ \Delta(T) = 0. \end{cases}$$

Note that the functions Σ_i and $(I + \Sigma_i R_1)^{-1}$, $i = 1, 2$ are bounded on $[0, T]$. Then a standard argument using the Gronwall inequality will show that $\Delta(\cdot) = 0$. The uniqueness of the solution to equation (3.20) is proved similarly. \square

5 Representations of optimal controls and value function

This section is going to give explicit formulas of the optimal controls and the value function, via the solutions to the Riccati equations (3.19), (3.20), and the MF-BSDE (3.21). Our first result can be stated as follows.

Theorem 5.1. *Let (H1)–(H2) hold and let $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$ be given. Let $\Sigma(\cdot)$ and $\Gamma(\cdot)$ be the unique solutions to the Riccati equations (3.19) and (3.20), respectively, and let $(\varphi(\cdot), \beta(\cdot))$ be the unique adapted solution to the MF-BSDE (3.21). Then the following MF-FSDE admits a unique solution $X(\cdot)$:*

$$(5.1) \quad \begin{cases} dX = \left\{ - (A + \Sigma Q)^\top X - [\bar{A} - \Sigma Q + \Gamma(Q + \bar{Q})]^\top \mathbb{E}[X] - Q\varphi - \bar{Q}\mathbb{E}[\varphi] \right\} ds \\ \quad + \left\{ [R_1(I + \Sigma R_1)^{-1}\Sigma - I]C^\top X + \left(-\bar{C}^\top - R_1(I + \Sigma R_1)^{-1}\Sigma C^\top \right. \right. \\ \quad \quad \left. \left. + (R_1 + \bar{R}_1)[I + \Sigma(R_1 + \bar{R}_1)]^{-1}\Sigma(C + \bar{C})^\top \right) \mathbb{E}[X] \right. \\ \quad \quad \left. - R_1(I + \Sigma R_1)^{-1}(\beta - \mathbb{E}[\beta]) - (R_1 + \bar{R}_1)[I + \Sigma(R_1 + \bar{R}_1)]^{-1}\mathbb{E}[\beta] \right\} dW, \\ X(t) = -[I + G\Sigma(t)]^{-1}G\{\varphi(t) - \mathbb{E}[\varphi(t)]\} - [I + (G + \bar{G})\Gamma(t)]^{-1}(G + \bar{G})\mathbb{E}[\varphi(t)], \end{cases}$$

and the unique optimal control of Problem (MF-BSLQ) for the terminal state ξ is given by

$$(5.2) \quad u = R_2^{-1}B^\top(X - \mathbb{E}[X]) + (R_2 + \bar{R}_2)^{-1}(B + \bar{B})^\top \mathbb{E}[X].$$

Proof. It is clear that (5.1) has a unique solution $X(\cdot)$. So we need only prove that $u(\cdot)$ defined by (5.2) is the unique optimal control of Problem (MF-BSLQ) for the terminal state ξ . To this end, we define

$$(5.3) \quad Y = -\Sigma(X - \mathbb{E}[X]) - \Gamma\mathbb{E}[X] - \varphi,$$

$$(5.4) \quad \begin{aligned} Z &= (I + \Sigma R_1)^{-1} \{ \Sigma C^\top (X - \mathbb{E}[X]) - (\beta - \mathbb{E}[\beta]) \} \\ &\quad + (I + \Sigma R_1 + \Sigma \bar{R}_1)^{-1} \{ \Sigma (C + \bar{C})^\top \mathbb{E}[X] - \mathbb{E}[\beta] \}. \end{aligned}$$

Then we have $Y(T) = \xi$ and

$$(5.5) \quad \mathbb{E}[Y] = -\Gamma \mathbb{E}[X] - \mathbb{E}[\varphi],$$

$$(5.6) \quad \mathbb{E}[Z] = (I + \Sigma R_1 + \Sigma \bar{R}_1)^{-1} \{ \Sigma (C + \bar{C})^\top \mathbb{E}[X] - \mathbb{E}[\beta] \}.$$

Also, from (5.4) and (5.6) we have

$$(5.7) \quad \begin{aligned} &Z + \Sigma R_1 Z + \Sigma \bar{R}_1 \mathbb{E}[Z] - \Sigma C^\top X - \Sigma \bar{C}^\top \mathbb{E}[X] + \beta \\ &= (I + \Sigma R_1)Z + \Sigma \bar{R}_1 \mathbb{E}[Z] - \Sigma C^\top X - \Sigma \bar{C}^\top \mathbb{E}[X] + \beta \\ &= \Sigma C^\top (X - \mathbb{E}[X]) - (\beta - \mathbb{E}[\beta]) + (I + \Sigma R_1) \mathbb{E}[Z] \\ &\quad + \Sigma \bar{R}_1 \mathbb{E}[Z] - \Sigma C^\top X - \Sigma \bar{C}^\top \mathbb{E}[X] + \beta \\ &= -\Sigma (C + \bar{C})^\top \mathbb{E}[X] + \mathbb{E}[\beta] + (I + \Sigma R_1 + \Sigma \bar{R}_1) \mathbb{E}[Z] \\ &= 0. \end{aligned}$$

Thus, making use of (3.19), (3.20), and (5.7), we have

$$\begin{aligned} dY &= -\dot{\Sigma}(X - \mathbb{E}[X])ds - \Sigma d(X - \mathbb{E}[X]) - \dot{\Gamma} \mathbb{E}[X]ds - \Gamma d\mathbb{E}[X] - d\varphi \\ &= -\dot{\Sigma}(X - \mathbb{E}[X])ds + \Sigma \left\{ (A + \Sigma Q)^\top (X - \mathbb{E}[X]) + Q(\varphi - \mathbb{E}[\varphi]) \right\} ds \\ &\quad - \Sigma \left\{ [R_1(I + \Sigma R_1)^{-1} \Sigma - I] C^\top X + \left(-\bar{C}^\top - R_1(I + \Sigma R_1)^{-1} \Sigma C^\top \right. \right. \\ &\quad \left. \left. + (R_1 + \bar{R}_1)(I + \Sigma R_1 + \Sigma \bar{R}_1)^{-1} \Sigma (C + \bar{C})^\top \right) \mathbb{E}[X] \right. \\ &\quad \left. - R_1(I + \Sigma R_1)^{-1} (\beta - \mathbb{E}[\beta]) - (R_1 + \bar{R}_1)(I + \Sigma R_1 + \Sigma \bar{R}_1)^{-1} \mathbb{E}[\beta] \right\} dW \\ &\quad - \dot{\Gamma} \mathbb{E}[X]ds + \Gamma \left\{ [A + \bar{A} + \Gamma(Q + \bar{Q})]^\top \mathbb{E}[X] + (Q + \bar{Q}) \mathbb{E}[\varphi] \right\} ds \\ &\quad - \left\{ (A + \Sigma Q) \varphi + [\bar{A} + \Gamma(Q + \bar{Q}) - \Sigma Q] \mathbb{E}[\varphi] + C(I + \Sigma R_1)^{-1} \beta \right. \\ &\quad \left. + [(C + \bar{C})(I + \Sigma R_1 + \Sigma \bar{R}_1)^{-1} - C(I + \Sigma R_1)^{-1}] \mathbb{E}[\beta] \right\} ds - \beta dW \\ &= \left\{ \left(-\dot{\Sigma} + \Sigma(A + \Sigma Q)^\top \right) (X - \mathbb{E}[X]) + \left(-\dot{\Gamma} + \Gamma[A + \bar{A} + \Gamma(Q + \bar{Q})]^\top \right) \mathbb{E}[X] \right. \\ &\quad \left. - A\varphi - \bar{A} \mathbb{E}[\varphi] - C(I + \Sigma R_1)^{-1} (\beta - \mathbb{E}[\beta]) - (C + \bar{C})(I + \Sigma R_1 + \Sigma \bar{R}_1)^{-1} \mathbb{E}[\beta] \right\} ds \\ &\quad - \left\{ \Sigma R_1(I + \Sigma R_1)^{-1} \Sigma C^\top (X - \mathbb{E}[X]) - \Sigma C^\top X - \Sigma \bar{C}^\top \mathbb{E}[X] \right. \\ &\quad \left. + \Sigma(R_1 + \bar{R}_1)(I + \Sigma R_1 + \Sigma \bar{R}_1)^{-1} \Sigma (C + \bar{C})^\top \mathbb{E}[X] - \Sigma R_1(I + \Sigma R_1)^{-1} (\beta - \mathbb{E}[\beta]) \right. \\ &\quad \left. - \Sigma(R_1 + \bar{R}_1)(I + \Sigma R_1 + \Sigma \bar{R}_1)^{-1} \mathbb{E}[\beta] + \beta \right\} dW \\ &= \left\{ \left(-A\Sigma + BR_2^{-1}B^\top + C(I + \Sigma R_1)^{-1} \Sigma C^\top \right) (X - \mathbb{E}[X]) + \left(-(A + \bar{A})\Gamma \right. \right. \\ &\quad \left. \left. + (B + \bar{B})(R_2 + \bar{R}_2)^{-1} (B + \bar{B})^\top + (C + \bar{C})(I + \Sigma R_1 + \Sigma \bar{R}_1)^{-1} \Sigma (C + \bar{C})^\top \right) \mathbb{E}[X] \right. \\ &\quad \left. - A\varphi - \bar{A} \mathbb{E}[\varphi] - C(I + \Sigma R_1)^{-1} (\beta - \mathbb{E}[\beta]) - (C + \bar{C})(I + \Sigma R_1 + \Sigma \bar{R}_1)^{-1} \mathbb{E}[\beta] \right\} ds \\ &\quad - \left\{ \Sigma R_1(I + \Sigma R_1)^{-1} \{ \Sigma C^\top (X - \mathbb{E}[X]) - (\beta - \mathbb{E}[\beta]) \} - \Sigma C^\top X - \Sigma \bar{C}^\top \mathbb{E}[X] + \beta \right. \\ &\quad \left. + \Sigma(R_1 + \bar{R}_1)(I + \Sigma R_1 + \Sigma \bar{R}_1)^{-1} \{ \Sigma (C + \bar{C})^\top \mathbb{E}[X] - \mathbb{E}[\beta] \} \right\} dW \end{aligned}$$

$$\begin{aligned}
&= \left\{ -A \left(\Sigma(X - \mathbb{E}[X]) + \Gamma \mathbb{E}[X] + \varphi \right) - \bar{A} \left(\Gamma \mathbb{E}[X] + \mathbb{E}[\varphi] \right) + B R_2^{-1} B^\top (X - \mathbb{E}[X]) \right. \\
&\quad + (B + \bar{B})(R_2 + \bar{R}_2)^{-1} (B + \bar{B})^\top \mathbb{E}[X] + C(I + \Sigma R_1)^{-1} \{ \Sigma C^\top (X - \mathbb{E}[X]) - (\beta - \mathbb{E}[\beta]) \} \\
&\quad \left. + (C + \bar{C})(I + \Sigma R_1 + \Sigma \bar{R}_1)^{-1} \{ \Sigma(C + \bar{C})^\top \mathbb{E}[X] - \mathbb{E}[\beta] \} \right\} ds \\
&\quad - \left\{ \Sigma R_1 (Z - \mathbb{E}[Z]) - \Sigma C^\top X - \Sigma \bar{C}^\top \mathbb{E}[X] + \beta + \Sigma(R_1 + \bar{R}_1) \mathbb{E}[Z] \right\} dW \\
&= \left\{ AY + \bar{A} \mathbb{E}[Y] + B(u - \mathbb{E}[u]) + (B + \bar{B}) \mathbb{E}[u] + C(Z - \mathbb{E}[Z]) + (C + \bar{C}) \mathbb{E}[Z] \right\} ds \\
&\quad - \left\{ \Sigma R_1 Z + \Sigma \bar{R}_1 \mathbb{E}[Z] - \Sigma C^\top X - \Sigma \bar{C}^\top \mathbb{E}[X] + \beta \right\} dW \\
&= \left\{ AY + \bar{A} \mathbb{E}[Y] + Bu + \bar{B} \mathbb{E}[u] + CZ + \bar{C} \mathbb{E}[Z] \right\} ds + Z dW.
\end{aligned}$$

Moreover, the first equation in (5.1) can be written as

$$\begin{aligned}
dX &= \left\{ -A^\top X - \bar{A}^\top \mathbb{E}[X] - Q \left(\Sigma(X - \mathbb{E}[X]) + \Gamma \mathbb{E}[X] + \varphi \right) - \bar{Q} \left(\Gamma \mathbb{E}[X] + \mathbb{E}[\varphi] \right) \right\} ds \\
&\quad + \left\{ -C^\top X - \bar{C}^\top \mathbb{E}[X] + R_1(I + \Sigma R_1)^{-1} \left(\Sigma C^\top (X - \mathbb{E}[X]) - (\beta - \mathbb{E}[\beta]) \right) \right. \\
&\quad \left. + (R_1 + \bar{R}_1)[I + \Sigma(R_1 + \bar{R}_1)]^{-1} \left(\Sigma(C + \bar{C})^\top \mathbb{E}[X] - \mathbb{E}[\beta] \right) \right\} dW \\
&= \left\{ -A^\top X - \bar{A}^\top \mathbb{E}[X] + QY + \bar{Q} \mathbb{E}[Y] \right\} ds + \left\{ -C^\top X - \bar{C}^\top \mathbb{E}[X] + R_1 Z + \bar{R}_1 \mathbb{E}[Z] \right\} dW.
\end{aligned}$$

From the second equation in (5.1), we see

$$(5.8) \quad \mathbb{E}[X(t)] = -[I + (G + \bar{G})\Gamma(t)]^{-1} (G + \bar{G}) \mathbb{E}[\varphi(t)],$$

$$(5.9) \quad X(t) - \mathbb{E}[X(t)] = -[I + G\Sigma(t)]^{-1} G \{ \varphi(t) - \mathbb{E}[\varphi(t)] \}.$$

(5.5) and (5.8) yield

$$[I + (G + \bar{G})\Gamma(t)] \mathbb{E}[X(t)] = -(G + \bar{G}) \mathbb{E}[\varphi(t)] = (G + \bar{G}) \{ \Gamma \mathbb{E}[X(t)] + \mathbb{E}[Y(t)] \},$$

from which follows

$$(5.10) \quad \mathbb{E}[X(t)] = (G + \bar{G}) \mathbb{E}[Y(t)].$$

Note that by (5.3) and (5.5),

$$Y(t) - \mathbb{E}[Y(t)] = -\Sigma(t) \{ X(t) - \mathbb{E}[X(t)] \} - \{ \varphi(t) - \mathbb{E}[\varphi(t)] \},$$

which, together with (5.9), yields

$$\begin{aligned}
[I + G\Sigma(t)] \{ X(t) - \mathbb{E}[X(t)] \} &= -G \{ \varphi(t) - \mathbb{E}[\varphi(t)] \} \\
&= G \left(\Sigma(t) \{ X(t) - \mathbb{E}[X(t)] \} + Y(t) - \mathbb{E}[Y(t)] \right),
\end{aligned}$$

from which follows

$$(5.11) \quad X(t) - \mathbb{E}[X(t)] = G \{ Y(t) - \mathbb{E}[Y(t)] \}.$$

Combining (5.10)–(5.11) we have

$$X(t) = GY(t) + \bar{G} \mathbb{E}[Y(t)].$$

Finally, observing that $u(\cdot)$ defined by (5.2) satisfies

$$R_2 u + \bar{R}_2 \mathbb{E}[u] - B^\top X - \bar{B}^\top \mathbb{E}[X] = 0,$$

we see that $(X(\cdot), Y(\cdot), Z(\cdot), u(\cdot))$ solves the optimality system (3.3)–(3.4). The result then follows immediately from Theorem 3.1. \square

We next present a formula for the value function of Problem (MF-BSLQ).

Theorem 5.2. *Let (H1)–(H2) hold. Then the value function of Problem (MF-BSLQ) is given by*

$$\begin{aligned} V(t, \xi) = & \mathbb{E} \left\{ \langle G[I + \Sigma(t)G]^{-1}(\varphi(t) - \mathbb{E}[\varphi(t)]), \varphi(t) - \mathbb{E}[\varphi(t)] \rangle \right. \\ & + \langle (G + \bar{G})[I + \Gamma(t)(G + \bar{G})]^{-1} \mathbb{E}[\varphi(t)], \mathbb{E}[\varphi(t)] \rangle \\ & + \int_t^T \left[\langle Q(\varphi - \mathbb{E}[\varphi]), \varphi - \mathbb{E}[\varphi] \rangle + \langle (Q + \bar{Q})\mathbb{E}[\varphi], \mathbb{E}[\varphi] \rangle \right. \\ & + \langle (I + R_1 \Sigma)^{-1} R_1 (\beta - \mathbb{E}[\beta]), \beta - \mathbb{E}[\beta] \rangle \\ & \left. \left. + \langle [I + (R_1 + \bar{R}_1)\Sigma]^{-1} (R_1 + \bar{R}_1) \mathbb{E}[\beta], \mathbb{E}[\beta] \rangle \right] ds \right\}. \end{aligned}$$

where $\Sigma(\cdot)$ and $\Gamma(\cdot)$ are the unique solutions to the Riccati equations (3.19) and (3.20), respectively, and $(\varphi(\cdot), \beta(\cdot))$ is the unique adapted solution to the MF-BSDE (3.21).

Proof. Let $(Y^*(\cdot), Z^*(\cdot), u^*(\cdot))$ be the optimal triple corresponding to the terminal state $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$, and let $X^*(\cdot)$ be the solution to MF-FSDE (3.1). According to Theorem 3.1, $(X^*(\cdot), Y^*(\cdot), Z^*(\cdot), u^*(\cdot))$ satisfies the optimality system (3.3)–(3.4). On the other hand, let $X(\cdot)$ be the solution to (5.1), and let $u(\cdot)$, $Y(\cdot)$, and $Z(\cdot)$ be defined by (5.2), (5.3), and (5.4), respectively. We recall from the proof of Theorem 5.1 that $(X(\cdot), Y(\cdot), Z(\cdot), u(\cdot))$ also satisfies the optimality system (3.3)–(3.4). By the uniqueness of optimal controls, we must have

$$(X^*(\cdot), Y^*(\cdot), Z^*(\cdot), u^*(\cdot)) = (X(\cdot), Y(\cdot), Z(\cdot), u(\cdot)).$$

Thus, the value $V(t, \xi)$ is equal to

$$\begin{aligned} J(t, \xi; u(\cdot)) = & \mathbb{E} \left\{ \int_t^T \left[\langle QY, Y \rangle + \langle \bar{Q}\mathbb{E}[Y], \mathbb{E}[Y] \rangle + \langle R_1 Z, Z \rangle + \langle \bar{R}_1 \mathbb{E}[Z], \mathbb{E}[Z] \rangle \right. \right. \\ & \left. \left. + \langle R_2 u, u \rangle + \langle \bar{R}_2 \mathbb{E}[u], \mathbb{E}[u] \rangle \right] ds + \langle GY(t), Y(t) \rangle + \langle \bar{G}\mathbb{E}[Y(t)], \mathbb{E}[Y(t)] \rangle \right\} \\ = & \mathbb{E} \left\{ \int_t^T \left[\langle Q(Y - \mathbb{E}[Y]), Y - \mathbb{E}[Y] \rangle + \langle (Q + \bar{Q})\mathbb{E}[Y], \mathbb{E}[Y] \rangle \right. \right. \\ & + \langle R_1 (Z - \mathbb{E}[Z]), Z - \mathbb{E}[Z] \rangle + \langle (R_1 + \bar{R}_1)\mathbb{E}[Z], \mathbb{E}[Z] \rangle \\ & + \langle R_2 (u - \mathbb{E}[u]), u - \mathbb{E}[u] \rangle + \langle (R_2 + \bar{R}_2)\mathbb{E}[u], \mathbb{E}[u] \rangle \left. \right] ds \\ & \left. + \langle G(Y(t) - \mathbb{E}[Y(t)]), Y(t) - \mathbb{E}[Y(t)] \rangle + \langle (G + \bar{G})\mathbb{E}[Y(t)], \mathbb{E}[Y(t)] \rangle \right\}. \end{aligned}$$

Noting that

$$\mathbb{E}[Y] = -\Gamma \mathbb{E}[X] - \mathbb{E}[\varphi],$$

$$\begin{aligned}
\mathbb{E}[Z] &= [I + \Sigma(R_1 + \bar{R}_1)]^{-1} \{ \Sigma(C + \bar{C})^\top \mathbb{E}[X] - \mathbb{E}[\beta] \}, \\
\mathbb{E}[u] &= (R_2 + \bar{R}_2)^{-1} (B + \bar{B})^\top \mathbb{E}[X], \\
Y - \mathbb{E}[Y] &= -\Sigma(X - \mathbb{E}[X]) - (\varphi - \mathbb{E}[\varphi]), \\
Z - \mathbb{E}[Z] &= (I + \Sigma R_1)^{-1} \{ \Sigma C^\top (X - \mathbb{E}[X]) - (\beta - \mathbb{E}[\beta]) \}, \\
u - \mathbb{E}[u] &= R_2^{-1} B^\top (X - \mathbb{E}[X]),
\end{aligned}$$

and using the fact that

$$(I + MN)^{-1}M = M(I + NM)^{-1}, \quad \forall M, N \in \overline{\mathbb{S}}_+^n,$$

it can be shown by a straightforward computation that

$$\begin{aligned}
(5.12) \quad & \mathbb{E} \int_t^T \left[\langle Q(Y - \mathbb{E}[Y]), Y - \mathbb{E}[Y] \rangle + \langle R_1(Z - \mathbb{E}[Z]), Z - \mathbb{E}[Z] \rangle + \langle R_2(u - \mathbb{E}[u]), u - \mathbb{E}[u] \rangle \right] ds \\
&= \mathbb{E} \int_t^T \left\{ \left\langle \left[\Sigma Q \Sigma + C(I + \Sigma R_1)^{-1} \Sigma R_1 \Sigma (I + R_1 \Sigma)^{-1} C^\top + B R_2^{-1} B^\top \right] (X - \mathbb{E}[X]), X - \mathbb{E}[X] \right\rangle \right. \\
&\quad + 2 \langle X - \mathbb{E}[X], \Sigma Q(\varphi - \mathbb{E}[\varphi]) \rangle - 2 \langle X - \mathbb{E}[X], C(I + \Sigma R_1)^{-1} \Sigma R_1 (I + \Sigma R_1)^{-1} (\beta - \mathbb{E}[\beta]) \rangle \\
&\quad \left. + \langle Q(\varphi - \mathbb{E}[\varphi]), \varphi - \mathbb{E}[\varphi] \rangle + \langle (I + R_1 \Sigma)^{-1} R_1 (I + \Sigma R_1)^{-1} (\beta - \mathbb{E}[\beta]), \beta - \mathbb{E}[\beta] \rangle \right\} ds,
\end{aligned}$$

and that

$$\begin{aligned}
(5.13) \quad & \int_t^T \left[\langle (Q + \bar{Q}) \mathbb{E}[Y], \mathbb{E}[Y] \rangle + \langle (R_1 + \bar{R}_1) \mathbb{E}[Z], \mathbb{E}[Z] \rangle + \langle (R_2 + \bar{R}_2) \mathbb{E}[u], \mathbb{E}[u] \rangle \right] ds \\
&= \int_t^T \left\{ \left\langle \left[\Gamma(Q + \bar{Q}) \Gamma + (C + \bar{C}) [I + \Sigma(R_1 + \bar{R}_1)]^{-1} \Sigma(R_1 + \bar{R}_1) \Sigma [I + (R_1 + \bar{R}_1) \Sigma]^{-1} (C + \bar{C})^\top \right. \right. \right. \\
&\quad \left. \left. + (B + \bar{B})(R_2 + \bar{R}_2)^{-1} (B + \bar{B})^\top \right] \mathbb{E}[X], \mathbb{E}[X] \right\rangle + 2 \langle \mathbb{E}[X], \Gamma(Q + \bar{Q}) \mathbb{E}[\varphi] \rangle \\
&\quad - 2 \langle \mathbb{E}[X], (C + \bar{C}) [I + \Sigma(R_1 + \bar{R}_1)]^{-1} \Sigma(R_1 + \bar{R}_1) [I + \Sigma(R_1 + \bar{R}_1)]^{-1} \mathbb{E}[\beta] \rangle \\
&\quad \left. + \langle (Q + \bar{Q}) \mathbb{E}[\varphi], \mathbb{E}[\varphi] \rangle + \langle [I + (R_1 + \bar{R}_1) \Sigma]^{-1} (R_1 + \bar{R}_1) [I + \Sigma(R_1 + \bar{R}_1)]^{-1} \mathbb{E}[\beta], \mathbb{E}[\beta] \rangle \right\} ds.
\end{aligned}$$

Observing that

$$\left\{ \begin{aligned} d\mathbb{E}[X] &= - \left\{ [A + \bar{A} + \Gamma(Q + \bar{Q})]^\top \mathbb{E}[X] + (Q + \bar{Q}) \mathbb{E}[\varphi] \right\} ds, \\ d(X - \mathbb{E}[X]) &= - \left\{ (A + \Sigma Q)^\top (X - \mathbb{E}[X]) + Q(\varphi - \mathbb{E}[\varphi]) \right\} ds \\ &\quad - \left\{ (I + R_1 \Sigma)^{-1} C^\top (X - \mathbb{E}[X]) + [I + (R_1 + \bar{R}_1) \Sigma]^{-1} (C + \bar{C})^\top \mathbb{E}[X] \right. \\ &\quad \left. + (I + R_1 \Sigma)^{-1} R_1 (\beta - \mathbb{E}[\beta]) + [I + (R_1 + \bar{R}_1) \Sigma]^{-1} (R_1 + \bar{R}_1) \mathbb{E}[\beta] \right\} dW, \end{aligned} \right.$$

we have by applying Itô's formula to $s \mapsto \langle \Sigma(s)(X(s) - \mathbb{E}[X(s)]), X(s) - \mathbb{E}[X(s)] \rangle$,

$$\begin{aligned}
(5.14) \quad & -\mathbb{E} \langle \Sigma(t) \{ X(t) - \mathbb{E}[X(t)] \}, X(t) - \mathbb{E}[X(t)] \rangle \\
&= \mathbb{E} \int_t^T \left\{ \left\langle \left[\dot{\Sigma} - (A + \Sigma Q) \Sigma - \Sigma(A + \Sigma Q)^\top + C(I + \Sigma R_1)^{-1} \Sigma (I + R_1 \Sigma)^{-1} C^\top \right] \right. \right. \\
&\quad \cdot (X - \mathbb{E}[X]), X - \mathbb{E}[X] \rangle - 2 \langle X - \mathbb{E}[X], \Sigma Q(\varphi - \mathbb{E}[\varphi]) \rangle \\
&\quad + 2 \langle X - \mathbb{E}[X], C(I + \Sigma R_1)^{-1} \Sigma R_1 (I + \Sigma R_1)^{-1} (\beta - \mathbb{E}[\beta]) \rangle \\
&\quad \left. + \langle (I + R_1 \Sigma)^{-1} R_1 \Sigma R_1 (I + \Sigma R_1)^{-1} (\beta - \mathbb{E}[\beta]), \beta - \mathbb{E}[\beta] \rangle \right\} ds
\end{aligned}$$

$$\begin{aligned}
& + \int_t^T \left\{ \langle (C + \bar{C})[I + \Sigma(R_1 + \bar{R}_1)]^{-1} \Sigma[I + (R_1 + \bar{R}_1)\Sigma]^{-1} (C + \bar{C})^\top \mathbb{E}[X], \mathbb{E}[X] \rangle \right. \\
& + 2 \langle \mathbb{E}[X], (C + \bar{C})[I + \Sigma(R_1 + \bar{R}_1)]^{-1} \Sigma(R_1 + \bar{R}_1)[I + \Sigma(R_1 + \bar{R}_1)]^{-1} \mathbb{E}[\beta] \rangle \\
& \left. + \langle [I + (R_1 + \bar{R}_1)\Sigma]^{-1} (R_1 + \bar{R}_1)\Sigma(R_1 + \bar{R}_1)[I + \Sigma(R_1 + \bar{R}_1)]^{-1} \mathbb{E}[\beta], \mathbb{E}[\beta] \rangle \right\} ds,
\end{aligned}$$

and by applying the integration by parts formula to $s \mapsto \langle \Gamma(s)\mathbb{E}[X(s)], \mathbb{E}[X(s)] \rangle$, we have

$$\begin{aligned}
(5.15) \quad & - \langle \Gamma(t)\mathbb{E}[X(t)], \mathbb{E}[X(t)] \rangle \\
& = \int_t^T \left\{ \left\langle \left(\dot{\Gamma} - [A + \bar{A} + \Gamma(Q + \bar{Q})]\Gamma - \Gamma[A + \bar{A} + \Gamma(Q + \bar{Q})]^\top \right) \mathbb{E}[X], \mathbb{E}[X] \right\rangle \right. \\
& \quad \left. - 2 \langle \mathbb{E}[X], \Gamma(Q + \bar{Q})\mathbb{E}[\varphi] \rangle \right\} ds.
\end{aligned}$$

Now adding equations (5.12), (5.13), (5.14) and (5.15) yields

$$\begin{aligned}
(5.16) \quad & V(t, \xi) = \mathbb{E} \left\{ \langle G(Y(t) - \mathbb{E}[Y(t)]), Y(t) - \mathbb{E}[Y(t)] \rangle + \langle (G + \bar{G})\mathbb{E}[Y(t)], \mathbb{E}[Y(t)] \rangle \right. \\
& \quad \left. + \langle \Sigma(t)\{X(t) - \mathbb{E}[X(t)]\}, X(t) - \mathbb{E}[X(t)] \rangle + \langle \Gamma(t)\mathbb{E}[X(t)], \mathbb{E}[X(t)] \rangle \right\} \\
& + \mathbb{E} \int_t^T \left\{ \langle Q(\varphi - \mathbb{E}[\varphi]), \varphi - \mathbb{E}[\varphi] \rangle + \langle (Q + \bar{Q})\mathbb{E}[\varphi], \mathbb{E}[\varphi] \rangle \right. \\
& \quad + \langle (I + R_1\Sigma)^{-1}R_1(\beta - \mathbb{E}[\beta]), \beta - \mathbb{E}[\beta] \rangle \\
& \quad \left. + \langle [I + (R_1 + \bar{R}_1)\Sigma]^{-1}(R_1 + \bar{R}_1)\mathbb{E}[\beta], \mathbb{E}[\beta] \rangle \right\} ds.
\end{aligned}$$

Recalling that

$$\mathbb{E}[Y] = -\Gamma\mathbb{E}[X] - \mathbb{E}[\varphi], \quad Y - \mathbb{E}[Y] = -\Sigma(X - \mathbb{E}[X]) - (\varphi - \mathbb{E}[\varphi]),$$

and noting that

$$\begin{aligned}
\mathbb{E}[X(t)] &= -[I + (G + \bar{G})\Gamma(t)]^{-1}(G + \bar{G})\mathbb{E}[\varphi(t)], \\
X(t) - \mathbb{E}[X(t)] &= -[I + G\Sigma(t)]^{-1}G\{\varphi(t) - \mathbb{E}[\varphi(t)]\},
\end{aligned}$$

we obtain

$$\begin{aligned}
& \mathbb{E} \left\{ \langle G(Y(t) - \mathbb{E}[Y(t)]), Y(t) - \mathbb{E}[Y(t)] \rangle + \langle (G + \bar{G})\mathbb{E}[Y(t)], \mathbb{E}[Y(t)] \rangle \right. \\
& \quad \left. + \langle \Sigma(t)\{X(t) - \mathbb{E}[X(t)]\}, X(t) - \mathbb{E}[X(t)] \rangle + \langle \Gamma(t)\mathbb{E}[X(t)], \mathbb{E}[X(t)] \rangle \right\} \\
& = \mathbb{E} \left\{ \langle G\{\Sigma(t)(X(t) - \mathbb{E}[X(t)]) + (\varphi(t) - \mathbb{E}[\varphi(t)])\}, \Sigma(t)(X(t) - \mathbb{E}[X(t)]) + (\varphi(t) - \mathbb{E}[\varphi(t)]) \rangle \right. \\
& \quad + \langle (G + \bar{G})\{\Gamma(t)\mathbb{E}[X(t)] + \mathbb{E}[\varphi(t)]\}, \Gamma(t)\mathbb{E}[X(t)] + \mathbb{E}[\varphi(t)] \rangle \\
& \quad \left. + \langle \Sigma(t)\{X(t) - \mathbb{E}[X(t)]\}, X(t) - \mathbb{E}[X(t)] \rangle + \langle \Gamma(t)\mathbb{E}[X(t)], \mathbb{E}[X(t)] \rangle \right\} \\
& = \mathbb{E} \left\{ \langle \Sigma(t)[I + G\Sigma(t)](X(t) - \mathbb{E}[X(t)]), X(t) - \mathbb{E}[X(t)] \rangle \right. \\
& \quad + 2 \langle G\Sigma(t)(X(t) - \mathbb{E}[X(t)]), \varphi(t) - \mathbb{E}[\varphi(t)] \rangle + \langle G(\varphi(t) - \mathbb{E}[\varphi(t)]), \varphi(t) - \mathbb{E}[\varphi(t)] \rangle \\
& \quad + \langle \Gamma(t)[I + (G + \bar{G})\Gamma(t)]\mathbb{E}[X(t)], \mathbb{E}[X(t)] \rangle + 2 \langle (G + \bar{G})\Gamma(t)\mathbb{E}[X(t)], \mathbb{E}[\varphi(t)] \rangle \\
& \quad \left. + \langle (G + \bar{G})\mathbb{E}[\varphi(t)], \mathbb{E}[\varphi(t)] \rangle \right\} \\
& = \mathbb{E} \left\{ \langle G[I + \Sigma(t)G]^{-1}(\varphi(t) - \mathbb{E}[\varphi(t)]), \varphi(t) - \mathbb{E}[\varphi(t)] \rangle \right. \\
& \quad \left. + \langle (G + \bar{G})[I + \Gamma(t)(G + \bar{G})]^{-1}\mathbb{E}[\varphi(t)], \mathbb{E}[\varphi(t)] \rangle \right\}.
\end{aligned}$$

Substitution of the above into (5.16) completes the proof. \square

References

- [1] M. Ait Rami, J. B. Moore, and X. Y. Zhou, *Indefinite stochastic linear quadratic control and generalized differential Riccati equation*, *SIAM J. Control Optim.*, 40 (2001), pp. 1296–1311.
- [2] D. Andersson and B. Djehiche, *A maximum principle for SDEs of mean-field type*, *Appl. Math. Optim.*, 63 (2011), pp. 341–356.
- [3] V. S. Borkar and K. S. Kumar, *McKean-Vlasov limit in portfolio optimization*, *Stochastic Anal. Appl.*, 28 (2010), pp. 884–906.
- [4] R. Buckdahn, B. Djehiche, and J. Li, *A general stochastic maximum principle for SDEs of mean-field type*, *Appl. Math. Optim.*, 64 (2011), pp. 197–216.
- [5] R. Buckdahn, B. Djehiche, J. Li, and S. Peng, *Mean-field backward stochastic differential equations: A limit approach*, *Ann. Probab.*, 37 (2009), pp. 1524–1565.
- [6] R. Buckdahn, J. Li, and S. Peng, *Mean-field backward stochastic differential equations and related partial differential equations*, *Stochastic Process. Appl.*, 119 (2009), pp. 3133–3154.
- [7] T. Chan, *Dynamics of the McKean-Vlasov equation*, *Ann. Probab.*, 22 (1994), pp. 431–441.
- [8] S. Chen, X. Li, and X. Y. Zhou, *Stochastic linear quadratic regulators with indefinite control weight costs*, *SIAM J. Control Optim.*, 36 (1998), pp. 1685–1702.
- [9] D. Crisan and J. Xiong, *Approximate McKean-Vlasov representations for a class of SPDEs*, *Stochastics*, 82 (2010), pp. 53–68.
- [10] D. A. Dawson, *Critical dynamics and fluctuations for a mean-field model of cooperative behavior*, *J. Statist. Phys.*, 31 (1983), pp. 29–85.
- [11] M. Kac, *Foundations of kinetic theory*, in: *Proc. 3rd Berkeley Symp. Math. Stat. Prob.*, 3 (1956), pp. 171–197.
- [12] X. Li, J. Sun, and J. Yong, *Mean-field stochastic linear quadratic optimal control problems: Closed-loop solvability*, *Probability, Uncertainty and Quantitative Risk*, 1:2 (2016), doi:10.1186/s41546-016-0002-3.
- [13] A. E. B. Lim and X. Y. Zhou, *Linear-quadratic control of backward stochastic differential equations*, *SIAM J. Control Optim.*, 40 (2001), pp. 450–474.
- [14] J. Ma, P. Protter, and J. Yong, *Solving forward-backward stochastic differential equations explicitly: A four-step scheme*, *Probab. Theory Related Fields*, 98 (1994), pp. 339–359.
- [15] J. Ma and J. Yong, *Forward-Backward Stochastic Differential Equations and Their Applications*, *Lecture Notes in Math.* 1702, Springer-Verlag, New York, 1999.
- [16] H. P. McKean, *A class of Markov processes associated with nonlinear parabolic equations*, *Proc. Natl. Acad. Sci. USA*, 56 (1966), pp. 1907–1911.
- [17] T. Meyer-Brandis, B. Oksendal, and X. Y. Zhou, *A mean-field stochastic maximum principle via Malliavin calculus*, *Stochastics*, 84 (2012), pp. 643–666.
- [18] J. Sun, *Mean-field stochastic linear quadratic optimal control problems: Open-loop solvabilities*, *ESAIM: COCV*, (2016), doi:10.1051/cocv/2016023.
- [19] J. Sun, X. Li, and J. Yong, *Open-loop and closed-loop solvabilities for stochastic linear quadratic optimal control problems*, *SIAM J. Control Optim.*, 54 (2016), pp. 2274–2308.

- [20] J. Sun and J. Yong, *Linear quadratic stochastic differential games: Open-loop and closed-loop saddle points*, *SIAM J. Control Optim.*, 52 (2014), pp. 4082–4121.
- [21] S. Tang, *General linear quadratic optimal stochastic control problems with random coefficients: linear stochastic Hamilton systems and backward stochastic Riccati equations*, *SIAM J. Control Optim.*, 42 (2003), pp. 53–75.
- [22] W. M. Wonham, *On a matrix Riccati equation of stochastic control*, *SIAM J. Control*, 6 (1968), pp. 681–697.
- [23] J. Yong, *Linear-quadratic optimal control problems for mean-field stochastic differential equations*, *SIAM J. Control Optim.*, 51 (2013), pp. 2809–2838.
- [24] J. Yong, *Linear-quadratic optimal control problems for mean-field stochastic differential equations – Time-consistent solutions*, *Transactions of the AMS*, (2015).
- [25] J. Yong and X. Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Springer-Verlag, New York, 1999.
- [26] D. Zhang, *Backward linear-quadratic stochastic optimal control and nonzero-sum differential game problem with random jumps*, *J. Syst. Sci. Complex*, 24 (2011), pp. 647–662.